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Closed rational trigonometric curves and surfaces

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ABSTRACT

In CAGD curves are described mostly by means of the combination of control points and basis functions. If we associate weights with basis functions and normalize them by their weighted sum, we obtain another set of basis functions that we call quotient bases. We show some common characteristics of curves defined by such quotient basis functions. Following this approach we specify the rational counterpart of the recently introduced cyclic basis, and provide a ready to use tool for control point based exact description of a class of closed rational trigonometric curves and surfaces. We also present the exact control point based description of some famous curves (Lemniscate of Bernoulli, Zhukovsky airfoil profile) and surfaces (Dupin cyclide and the smooth transition between the Boy surface and the Roman surface of Steiner) to illustrate the usefulness of the proposed tool.

1. Introduction

In CAGD the most widespread description form of curves is

$$\begin{gathered}\mathbf{g}(u) = \sum_{j=0}^{n} F_j(u)\,\mathbf{d}_j \\ F_j : [a, b] \to \mathbb{R}, \qquad u \in [a, b] \subset \mathbb{R}, \qquad \mathbf{d}_j \in \mathbb{R}^{\delta}, \quad \delta \geq 2\end{gathered} \tag{1}$$

where $\mathbf{d}_j$ are called control points, the polygon determined by vertices $\mathbf{d}_j$ is the control polygon and F_j are basis functions. This means that a curve can be specified by just a few control points which is advantageous from data storage and transmission point of view, moreover, if the functions F_j are properly chosen the resulting curve mimics the shape of the control polygon, that is the control polygon provides an intuitive tool for the designer. A comprehensive study on requirements imposed upon basis functions can be found in [1].

A subset of (1) that is often used for design purposes possesses endpoint interpolation, i.e. the curve starts at $\mathbf{d}_0$ and terminates at $\mathbf{d}_n$. The most well known representative of this subset is the Bézier curve and its rational counterpart a comprehensive study of which can be found, e.g. in [2,3]. There are other generalizations of the Bézier curve, such as C-Bézier (cf. [4,5]) and H-Bézier (cf. [6]) curves.

Another important subset of curves (1) is formed by spline curves when the domain $[a, b]$ is subdivided into smaller intervals and the basis functions are defined piecewisely on these ranges guaranteeing a certain order of continuity at the joints. The most popular spline basis functions are normalized *B*-spline functions and their rational generalization, the NURBS functions, for a comprehensive study of which cf. [3]. There are other spline basis functions that comprise

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polynomials and trigonometric functions (cf. [7–9]), or hyperbolic functions [10] or both trigonometric and hyperbolic (cf. [11]).

Recently, in the space of trigonometric polynomials a new type of basis functions – called cyclic basis – has been published (cf. [12]) using which closed curves/surfaces can be described by control points without redundancy and with the advantage of C^{∞} continuity at all points of the curve/surface.

The main objective of this paper is to provide an exact formula for control point based description of closed curves/surfaces the coordinate functions of which are rational trigonometric polynomials of finite degree.

The rest of the paper is organized as follows. Section 2 outlines the common characteristics of quotient basis functions and objects defined by them. Section 3 briefly reviews cyclic curves/surfaces and lists those properties of them that will be used in the subsequent sections. Section 4 introduces a control point and weight based modeling tool for the description of closed rational trigonometric curves and surfaces by means of the rational counterpart of cyclic curves and surfaces. Section 5 provides several applications that emphasize the usefulness of the proposed rational trigonometric basis.

2. Quotient basis functions

Let us consider the system of linearly independent, non-negative, normalized functions $F = \{F_i : [a, b] \to \mathbb{R}\}_{i=0}^{n}$ and scalar values $w_0, w_1, \dots, w_n \in \mathbb{R}$ that satisfy the condition $\sum_{i=0}^{n} w_i^2 \neq 0$.

If scalar values w_i are also non-negative, then function system $R = \{R_i : [a, b] \to \mathbb{R}\}_{i=0}^{n}$

$$R_i(u) = \frac{w_i F_i(u)}{\sum_{j=0}^{n} w_j F_j(u)}, \quad u \in [a, b] \tag{2}$$

inherits the properties of F, i.e. the system R is also non-negative, normalized and linearly independent.

By means of these quotient functions we can generate curves of type (1)

$$\begin{aligned} \mathbf{g}(u) &= \sum_{i=0}^{n} R_i(u)\, \mathbf{b}_i \\ &= \sum_{i=0}^{n} \frac{w_i F_i(u)}{\sum_{j=0}^{n} w_j F_j(u)} \mathbf{b}_i, \quad u \in [a, b], \mathbf{b}_i \in \mathbb{R}^{\delta}, (\delta \geq 2), \end{aligned} \tag{3}$$

where scalar values w_i are called weights. It is obvious that the weights are determined up to a positive scaling factor, i.e. weights w_i and λw_i, $0 < \lambda \in \mathbb{R}$ specify the same curve. If all weights are equal we obtain the original set of basis functions F as a special case.

Curve (3) can also be considered as the central projection of the curve

$$\mathbf{g}^w(u) = \sum_{i=0}^{n} F_i(u) \begin{bmatrix} w_i \mathbf{b}_i \\ w_i \end{bmatrix}, \quad u \in [a, b]$$

in the $\delta + 1$ dimensional space from the origin onto the δ dimensional hyperplane $w = 1$ (assuming that the last coordinate of space $\mathbb{R}^{\delta+1}$ is denoted by w). Curve $\mathbf{g}^w$ is called the pre-image of curve $\mathbf{g}$.

This central projection concept facilitates to study the properties of curve $\mathbf{g}$. Curve $\mathbf{g}$ inherits all properties of $\mathbf{g}^w$ that are invariant under central projection, such as continuity, incidence, collinearity and variation diminishing. Curve (3) is closed for the projective transformation of its control points, i.e. the curve determined by the transformed control points coincides with the transformed curve. The transformation has to be performed in the pre-image space, therefore not only control points but weights will also be altered. (Note, that curves defined by the original set of basis functions $F = \{F_i\}_{i=0}^{n}$ are closed only for the affine transformation of their control points.)

Well known representatives of quotient basis (2) are the rational Bernstein and B-spline functions. Curves generated by them are widespread, and the rational B-spline curve (also called NURBS curve) has become the de facto standard of curve description in the geometric modeling core of nowadays CAD systems. The application of rational curves and surfaces in geometric modeling was pioneered in [13–16]. Properties of rational Bézier and B-spline curves and surfaces can be found in standard books, like Hoschek, Lasser [3], Farin [17,18], Piegl, Tiller [19,20].

In this paper we show that several properties of rational curves/surfaces hold for this generalized case, then we specify the rational counterpart of cyclic curves/surfaces and, finally, provide some examples for their application.

2.1. Shape modification

A definite drawback of curves defined by quotient basis function is that their derivatives (especially higher ones) become more complicated. In return, however, we gain a new shape modification tool, since using the weights the influence of individual control points can be increased/decreased.

The alteration of any of the defining data of curve (1) implies the modification of the shape of the curve. A basic question is that what kind of curve is traced by a selected point of the curve (1) when at least one of its defining data is modified. We will refer to this traced curve as the path of the selected point.

The simplest and most effective shape modification tool of curve (1) is control point repositioning. If control point $\mathbf{d}_j$ is shifted by the vector $\mathbf{v}$, we obtain the curve

$$\sum_{i=0}^{n} F_i(u)\,\mathbf{d}_i + F_j(u)\,\mathbf{v} = \mathbf{g}(u) + F_j(u)\,\mathbf{v},$$

i.e. paths of points of curve (1) are straight lines parallel to the shift vector.

If the basis functions are quotient functions of type (2), then weights provide an additional shape modification tool. Paths of points of rational B-spline curves has been studied in [21,19]. Those results are valid for the generic case as well but the proof is different (the central projection – projective geometry – concept has to be applied).

Theorem 1. *If the weight w_j of curve (3) is modified, points of the curve move along straight lines that pass through the control point $\mathbf{b}_j$.*

Proof. Modification of weight w_j results in the pre-image space a translation of the control point $[\, w_j\mathbf{b}_j \quad w_j \,]^T$ along its position vector. Therefore, points of the curve $\mathbf{g}^w$ are shifted parallel to the translation vector. Projecting these parallel straight lines from the origin onto the hyperplane $w = 1$, we obtain straight lines that pass through the control point $\mathbf{b}_j$, since it is their common vanishing point. $\square$

Theorem 2. *If weights w_j and w_k of curve (3) are simultaneously modified, points of the curve move in planes that pass through the control points $\mathbf{b}_j$ and $\mathbf{b}_k$.*

Proof. For plane curves the statement is trivially true. For the case of spatial curves we utilize the central projection concept. Simultaneous modification of weights w_j and w_k in the pre-image space results the translation of points of the curve $\mathbf{g}^w$ parallel to vectors that are linear combinations of position vectors $[\, w_j\mathbf{b}_j \quad w_j \,]^T$ and $[\, w_k\mathbf{b}_k \quad w_k \,]^T$. Therefore, the modified points will be in such planes that are parallel to the vectors $[\, w_j\mathbf{b}_j \quad w_j \,]^T$ and $[\, w_k\mathbf{b}_k \quad w_k \,]^T$. Projecting these from the origin onto the hyperplane $w = 1$, we obtain planes that pass through the joining line of control points $\mathbf{b}_j$ and $\mathbf{b}_k$, since this line is their common vanishing line. $\square$

Weights can be varied in the range $[0, \infty)$ in order to preserve convex hull property and to avoid singularities. Hereafter, we denote the elements of the family of curves that is obtained by the modification of weight w_k by $\mathbf{g}(u, w_k)$.

Theorem 3. *When the weight w_k is modified, the cross ratio of collinear points $\mathbf{q}_0 = \mathbf{g}(u, 0)$, $\mathbf{q}_1 = \mathbf{g}(u, 1)$, $\mathbf{q} = \mathbf{g}(u, w_k)$ and $\mathbf{d}_k$ is $(\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}, \mathbf{d}_k) = w_k$.*

Proof. Points of the curve corresponding to the weights $w_k = 0$ and $w_k = 1$ are

$$\mathbf{q}_0 = \frac{\sum\limits_{i=0, i\neq k}^{n} w_i F_i(u)\,\mathbf{d}_i}{\sum\limits_{j=0, j\neq k}^{n} w_j F_j(u)} \quad \text{and}$$

$$\mathbf{q}_1 = \frac{\sum\limits_{i=0, i\neq k}^{n} w_i F_i(u)\,\mathbf{d}_i}{\sum\limits_{j=0, j\neq k}^{n} w_j F_j(u) + F_k(u)} + \frac{F_k(u)\,\mathbf{d}_k}{\sum\limits_{j=0, j\neq k}^{n} w_j F_j(u) + F_k(u)}$$

respectively. $\mathbf{q}_1$ is an inner point of the straight line segment $\mathbf{q}_0, \mathbf{d}_k$, thus it can be generated as a convex combination of the points $\mathbf{q}_0$ and $\mathbf{d}_k$. The combining coefficient is

$$\alpha = \frac{F_k(u)}{\sum\limits_{j=0, j\neq k}^{n} w_j F_j(u) + F_k(u)},$$

since utilizing that the function system F is normalized, we obtain

$$(1-\alpha)\,\mathbf{q}_0 + \alpha \mathbf{d}_k = \mathbf{q}_1.$$

The point $\mathbf{q} = \mathbf{g}(u, w_k)$ that corresponds to a generic w_k is

$$\mathbf{q} = \frac{\sum_{i=0, i\neq k}^{n} w_i F_i(u)\,\mathbf{d}_i}{\sum_{j=0}^{n} w_j F_j(u)} + \frac{w_k F_k(u)\,\mathbf{d}_k}{\sum_{j=0}^{n} w_j F_j(u)}.$$

This point can also be produced as a convex combination of points $\mathbf{q}_0$ and $\mathbf{d}_k$ with the coefficient

$$\beta = \frac{w_k F_k(u)}{\sum_{j=0}^{n} w_j F_j(u)}, \tag{4}$$

since $(1-\beta)\mathbf{q}_0 + \beta\mathbf{d}_k = \mathbf{q}$. Therefore, the cross ratio of points $\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}, \mathbf{d}_p$ is

$$\left(\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}, \mathbf{d}_p\right) = \frac{\left(\mathbf{q}_0, \mathbf{q}_1, \mathbf{d}_p\right)}{\left(\mathbf{q}_0, \mathbf{q}, \mathbf{d}_p\right)} = \frac{1-\alpha}{\alpha}\frac{\beta}{1-\beta} = w_k. \quad \square$$

Corollary 4. *An arbitrary point $\mathbf{g}(\widehat{u})$ of the curve $\mathbf{g}$ can be taken into any point of the straight line segment $\mathbf{q}_0 = \mathbf{g}(\widehat{u}, 0)$, $\mathbf{d}_k$ (except the point $\mathbf{d}_k$) by means of the modification of the weight w_k. Any point $\mathbf{p} \neq \mathbf{d}_k$ of the segment $\mathbf{q}_0$, $\mathbf{d}_k$ can be written in the form $\mathbf{p} = (1-\beta)\mathbf{q}_0 + \beta\mathbf{d}_k$. Using the formula (4) we obtain*

$$w_k = \frac{\beta \sum_{i=0, i\neq k}^{n} w_i F_i\left(\hat{u}\right)}{(1-\beta)\, F_k\left(\hat{u}\right)}.$$

Analogous results can be derived for tensor product surfaces specified by quotient basis functions (2).

3. Cyclic curves and surfaces

In [12] a new basis – called cyclic basis – is given for the description of closed curves and surfaces. The cyclic basis of the vector space

$$\mathcal{V}_n = \langle 1, \cos(u), \sin(u), \ldots, \cos(nu), \sin(nu)\rangle$$

of trigonometric polynomials of finite degree at most $n \geq 1$ is

$$\begin{aligned} C &= \left\{C_{i,n}(u) : u \in [-2\pi, 0]\right\}_{i=0}^{2n} \\ &= \left\{\frac{c_n}{2^n}\left(1+\cos\left(u+i\lambda_n\right)\right)^n : u \in [-2\pi, 0]\right\}_{i=0}^{2n}, \end{aligned} \tag{5}$$

where $\lambda_n = 2\pi/(2n+1)$ and constant

$$c_n = \frac{2^{2n}}{(2n+1)\binom{2n}{n}}$$

fulfills the recursion

$$\begin{cases} c_1 = \dfrac{2}{3}, \\ c_n = \dfrac{2n}{2n+1} c_{n-1}, \quad n \geq 2. \end{cases}$$

Note, the basis (5) consists of 2π-periodic functions, thus we can study the properties of this basis on any interval of length 2π.

Using these basis functions, the so called cyclic curves and surfaces can be specified.

Definition 5 (*Cyclic Curve*)**.** The curve

$$\mathbf{a}_n(u) = \sum_{i=0}^{2n} C_{i,n}(u)\,\mathbf{d}_i, \quad u \in [-2\pi, 0], \tag{6}$$

is called cyclic curve of degree $n \geq 1$ that is uniquely determined by its control polygon

$$\mathbf{D}_n = [\mathbf{d}_0, \mathbf{d}_1, \ldots, \mathbf{d}_{2n}] = [\mathbf{d}_i]_{i=0}^{2n} \in \mathcal{M}_{1,2n+1}\left(\mathbb{R}^\delta\right), \quad \delta \geq 2,$$

and basis functions (5).

These curves have the following advantageous properties:

- singularity free parametrization (the curve/surface is of C^∞ continuity at all regular points and at singular points non-vanishing left and right derivatives exist);
- convex hull property;
- cyclic symmetry (the shape of the curve does not change when its control points are cyclically permuted);
- closure for the affine transformation of their control points;
- pseudo local controllability;
- variation diminishing;
- an efficient closed formula for the degree elevation from n to $n+r$, $r \geq 1$ that results in a sequence of control polygons that converges to the cyclic curve.

Definition 6 (*Cyclic Surface*)**.** The surface

$$\mathbf{s}_{n,m}(u, v) = \sum_{i=0}^{2n}\sum_{j=0}^{2m} \mathbf{d}_{ij} C_{i,n}(u)\, C_{j,m}(v), \quad (u, v) \in [-2\pi, 0] \times [-2\pi, 0], \tag{7}$$

is called cyclic surface of degree (n, m) $(n \geq 1, m \geq 1)$ that is uniquely determined by its control net

$$\mathbf{D}_{n,m} = \left[\mathbf{d}_{ij}\right]_{i=0,j=0}^{2n,2m} \in \mathcal{M}_{2n+1,2m+1}\left(\mathbb{R}^3\right)$$

and basis functions $C_{i,n}(u)$ and $C_{j,m}(v)$ defined by (5).

With the exception of variation diminishing all properties of cyclic curves are inherited by cyclic surfaces.

Paper [22] has specified control point configurations that result an exact description of those closed curves and surfaces the coordinate functions of which are (separable) trigonometric polynomials of finite degree. The main theoretical results of this article, that we will use in the subsequent sections, are the lemmas and theorems formulated below.

A wide range of closed curves (like ellipses (circles), epi- and hypocycloids, Lissajous curves, torus knots, foliums, etc.) can be described in the form

$$\begin{cases} \mathbf{x} : [-2\pi, 0] \to \mathbb{R}^\delta, \quad \delta \geq 2, \\ \mathbf{x}(u) = \begin{bmatrix} x_1(u) & x_2(u) & \cdots & x_\delta(u) \end{bmatrix}^T, \end{cases} \tag{8}$$

where

$$\begin{aligned} x_l(u) &\doteq x_l\left(u, \left\{\left(\alpha_p^l, \psi_p^l\right)\right\}_{p\in P_l}, \left\{\left(\beta_q^l, \varphi_q^l\right)\right\}_{q\in Q_l}\right) \\ &= \sum_{p\in P_l} \alpha_p^l \cos\left(pu + \psi_p^l\right) + \sum_{q\in Q_l} \beta_q^l \sin\left(qu + \varphi_q^l\right), \quad (l = 1, 2, \ldots, \delta), \end{aligned} \tag{9}$$

and $P_l, Q_l \subset \mathbb{N}$, $\alpha_p^l, \beta_q^l, \psi_p^l, \varphi_q^l \in \mathbb{R}$. Article [22] gave control point configurations that exactly describe such kind of parametric closed curves. All results on the class of curves (8) are based on the next lemma.

Lemma 7 (*Core Property of Class* (8))**.** *Consider the closed curve* (8) *and let*

$$n \geq n_{\min} = \max\left\{e \mid e \in \bigcup_{l=1}^{\delta}(P_l \cup Q_l)\right\}.$$

Introduce control points

$$\mathbf{d}_i = \begin{bmatrix} x_1(-i\lambda_n) & x_2(-i\lambda_n) & \cdots & x_\delta(-i\lambda_n) \end{bmatrix}^T, \quad i = 0, 1, \ldots, 2n, \tag{10}$$

and the cyclic curve

$$\begin{aligned}\mathbf{a}_n(u) &= \sum_{i=0}^{2n} C_{i,n}(u)\,\mathbf{d}_i \\ &= \begin{bmatrix} a_1(u) & a_2(u) & \cdots & a_\delta(u)\end{bmatrix}^T, \quad u \in [-2\pi, 0]\end{aligned}$$

defined by these control points. In this case the lth ($l = 1, 2, \ldots, \delta$) coordinate function of $\mathbf{a}_n$ *can be expressed as*

$$a_l(u) = \frac{1}{\binom{2n}{n}}\left(\sum_{p\in P_l}\alpha_p^l\binom{2n}{n-p}\cos\left(pu+\psi_p^l\right) + \sum_{q\in Q_l}\beta_q^l\binom{2n}{n-q}\sin\left(qu+\varphi_q^l\right)\right). \tag{11}$$

Vice versa, if the coordinate functions of a cyclic curve are of the form (11), then its control points are exactly the points (10).

The direct application of Lemma 7 provides a ready to use tool for the control point based exact description of curves (8) by means of cyclic curves. The method is formulated in the next theorem.

Theorem 8 (*Exact Description of Curve Class (8)*). *Consider the closed curve*

$$\widetilde{\mathbf{x}}(u) = \begin{bmatrix}\widetilde{x}_1(u) & \widetilde{x}_2(u) & \cdots & \widetilde{x}_\delta(u)\end{bmatrix}^T, \quad \delta \ge 2, u \in [-2\pi, 0],$$

the coordinate functions of which are

$$\widetilde{x}_l(u) = \sum_{p\in P_l}\widetilde{\alpha}_p^l\cos\left(pu+\widetilde{\psi}_p^l\right) + \sum_{q\in Q_l}\widetilde{\beta}_q^l\sin\left(qu+\widetilde{\varphi}_q^l\right), \quad (l = 1, 2, \ldots, \delta), \tag{12}$$

where $P_l, Q_l \subset \mathbb{N}$ *and* $\widetilde{\alpha}_p^l, \widetilde{\beta}_q^l, \widetilde{\psi}_p^l, \widetilde{\varphi}_q^l \in \mathbb{R}$. *Let*

$$n \ge n_{\min} = \max\left\{e \mid e \in \bigcup_{l=1}^{\delta}(P_l \cup Q_l)\right\}$$

and $r \in \mathbb{N}$. *If we set constants that are in the representation of curve* $\mathbf{x}$ *in Lemma 7 as*

$$\alpha_p^l(n) = \frac{\binom{2n}{n}}{\binom{2n}{n-p}}p^r\widetilde{\alpha}_p^l, \qquad \psi_p^l(n) \equiv \widetilde{\psi}_p^l + \frac{r\pi}{2}, \quad p \in P_l, \tag{13}$$

$$\beta_q^l(n) = \frac{\binom{2n}{n}}{\binom{2n}{n-q}}q^r\widetilde{\beta}_q^l, \qquad \varphi_q^l(n) \equiv \widetilde{\varphi}_q^l + \frac{r\pi}{2}, \quad q \in Q_l, \tag{14}$$

then for all $n \ge n_{\min}$ *the cyclic curve generated by control points (10) describes exactly the rth order derivative of the curve* $\widetilde{\mathbf{x}}$, *i.e. the lth coordinate function of cyclic curve* $\mathbf{a}_n$ *fulfills the equality*

$$\begin{aligned}a_l(u) = \frac{\mathrm{d}^r}{\mathrm{d}u^r}\widetilde{x}^l(u) &= \sum_{p\in P_l}p^r\widetilde{\alpha}_p^l\cos\left(pu+\widetilde{\psi}_p^l+\frac{r\pi}{2}\right) + \sum_{q\in Q_l}q^r\widetilde{\beta}_q^l\sin\left(qu+\widetilde{\varphi}_q^l+\frac{r\pi}{2}\right)\\ &\forall u \in [-2\pi, 0], (l = 1, 2, \ldots, \delta).\end{aligned}$$

We consider those closed surfaces that are the members of a general class of surfaces which can be expressed in the form

$$\begin{cases}\mathbf{y} : [-2\pi, 0]^\delta \to \mathbb{R}^{\delta+\kappa}, \quad \delta \ge 2,\ \kappa \ge 1,\\ \mathbf{y}(\mathbf{u}) = \begin{bmatrix} y_1(\mathbf{u}) & y_2(\mathbf{u}) & \cdots & y_{\delta+\kappa}(\mathbf{u})\end{bmatrix}^T, \quad \mathbf{u} = (u_1, u_2, \ldots, u_\delta),\end{cases} \tag{15}$$

where

$$\begin{aligned}y_l(\mathbf{u}) &\doteq y_l\left(\mathbf{u}, \left\{\left\{\left(\alpha_p^{l,w,j}, \psi_p^{l,w,j}\right)\right\}_{p\in P_{l,w,j}}, \left\{\left(\beta_q^{l,w,j}, \varphi_q^{l,w,j}\right)\right\}_{q\in Q_{l,w,j}}\right\}_{w=1,j=1}^{m_l,\delta}\right)\\ &= \sum_{w=1}^{m_l}\prod_{j=1}^{\delta}\left(\sum_{p\in P_{l,w,j}}\alpha_p^{l,w,j}\cos\left(pu_j+\psi_p^{l,w,j}\right) + \sum_{q\in Q_{l,w,j}}\beta_q^{l,w,j}\sin\left(qu_j+\varphi_q^{l,w,j}\right)\right), \quad l = 1, 2, \ldots, \delta+\kappa,\end{aligned}$$

and $P_{l,w,j}, Q_{l,w,j} \subset \mathbb{N}$, $m_l \in \mathbb{N}\setminus\{0\}$ and $\alpha_p^{l,w,j}, \beta_q^{l,w,j}, \psi_p^{l,w,j}, \varphi_q^{l,w,j} \in \mathbb{R}$. Members of this class are, e.g. surfaces of revolution obtained by rotating a curve of type (8) about an axis and the Roman surface of Steiner. Paper [22] also proposed control point configurations for the exact description of such kind of closed surfaces by means of cyclic surfaces. The results concerning the class of surfaces (15) are based on the next lemma.

Lemma 9 (*Core Property of Class* (15)). *Consider the closed surface* (15). *Let*

$$n_j \geq n_{\min}^j = \max\left\{e_j \mid e_j \in \bigcup_{l=1}^{\delta+\kappa}\bigcup_{w=1}^{m_l}\left(P_{l,w,j} \cup Q_{l,w,j}\right)\right\}, \quad (j = 1, 2, \ldots, \delta)$$

and consider the control net

$$\left[\mathbf{d}_{i_1 i_2 \cdots i_\delta}\right]_{i_1=0, i_2=0, \ldots, i_\delta=0}^{2n_1, 2n_2, \ldots, 2n_\delta} \in \mathcal{M}_{2n_1+1, 2n_2+1, \ldots, 2n_\delta+1}\left(\mathbb{R}^{\delta+\kappa}\right)$$

the elements of which are given by

$$\mathbf{d}_{i_1 i_2 \cdots i_\delta} = \left[y_1\left(\mathbf{u}^*_{i_1 i_2 \cdots i_\delta}\right) \quad y_2\left(\mathbf{u}^*_{i_1 i_2 \cdots i_\delta}\right) \quad \cdots \quad y_{\delta+\kappa}\left(\mathbf{u}^*_{i_1 i_2 \cdots i_\delta}\right)\right]^T, \tag{16}$$

where

$$\mathbf{u}^*_{i_1 i_2 \cdots i_\delta} = \left(-i_1\lambda_{n_1}, -i_2\lambda_{n_2}, \ldots, -i_\delta\lambda_{n_\delta}\right) \in \mathbb{R}^\delta.$$

Consider also the cyclic surface $\mathbf{s}_{n_1,n_2,\ldots,n_\delta} : [-2\pi, 0]^\delta \to \mathbb{R}^{\delta+\kappa}$ *of degree* $(n_1, n_2, \ldots, n_\delta)$ *defined by the tensor product*

$$\begin{aligned}
\mathbf{s}_{n_1,n_2,\ldots,n_\delta}(\mathbf{u}) &= \sum_{i_1=0}^{2n_1}\sum_{i_2=0}^{2n_2}\cdots\sum_{i_\delta=0}^{2n_\delta} C_{i_1,n_1}(u_1)\, C_{i_2,n_2}(u_2)\cdots C_{i_\delta,n_\delta}(u_\delta)\,\mathbf{d}_{i_1 i_2 \cdots i_\delta} \\
&= \left(\prod_{j=1}^{\delta}\frac{c_{n_j}}{2^{n_j}}\right)\sum_{i_1=0}^{2n_1}\sum_{i_2=0}^{2n_2}\cdots\sum_{i_\delta=0}^{2n_\delta}\left(1+\cos\left(u_1+i_1\lambda_{n_1}\right)\right)^{n_1} \\
&\quad \cdot \left(1+\cos\left(u_2+i_2\lambda_{n_2}\right)\right)^{n_2}\cdots\left(1+\cos\left(u_\delta+i_\delta\lambda_{n_\delta}\right)\right)^{n_\delta}\mathbf{d}_{i_1 i_2 \cdots i_\delta} \\
&= \left[s_1(\mathbf{u}) \quad s_2(\mathbf{u}) \quad \cdots \quad s_{\delta+\kappa}(\mathbf{u})\right]^T, \quad \mathbf{u} = (u_1, u_2, \ldots, u_\delta) \in [-2\pi, 0]^\delta.
\end{aligned}$$

With these settings the lth ($l = 1, 2, \ldots, \delta+\kappa$) *coordinate function of surface* $\mathbf{s}_{n_1,n_2,\ldots,n_\delta}$ *can be expressed in the form*

$$s_l(\mathbf{u}) = \sum_{w=1}^{m_l}\prod_{j=1}^{\delta}\frac{1}{\binom{2n_j}{n_j}}\left(\sum_{p\in P_{l,w,j}}\alpha_p^{l,w,j}\binom{2n_j}{n_j-p}\cos\left(pu_j+\psi_p^{l,w,j}\right)+\sum_{q\in Q_{l,w,j}}\beta_q^{l,w,j}\binom{2n_j}{n_j-q}\sin\left(qu_j+\varphi_q^{l,w,j}\right)\right). \tag{17}$$

The inverse statement is also valid, i.e. if the coordinate functions of a cyclic surface of degree $n_1, n_2, \ldots, n_\delta$ *are of the form* (17), *then the vertices of its control net are exactly the points* (16).

Finally, the direct application of Lemma 9 results the next theorem which provides control point configurations for the exact description of surfaces of type (15) by means of cyclic surfaces.

Theorem 10 (*Exact Description of Surface Class* (15)). *Let* $\delta = 2$ *and* $\kappa = 1$ *in Lemma* 9. *Consider also the closed surface*

$$\widetilde{\mathbf{y}}(u_1, u_2) = \left[\widetilde{y}_1(u_1, u_2) \quad \widetilde{y}_2(u_1, u_2) \quad \widetilde{y}_3(u_1, u_2)\right]^T, \quad (u_1, u_2) \in [-2\pi, 0] \times [-2\pi, 0],$$

the coordinate functions of which are

$$\widetilde{y}_l(u_1, u_2) = \sum_{w=1}^{m_l}\prod_{j=1}^{2}\left(\sum_{p\in P_{l,w,j}}\widetilde{\alpha}_p^{l,w,j}\cos\left(pu_j+\widetilde{\psi}_p^{l,w,j}\right)+\sum_{q\in Q_{l,w,j}}\widetilde{\beta}_q^{l,w,j}\sin\left(qu_j+\widetilde{\varphi}_q^{l,w,j}\right)\right), \tag{18}$$

($l = 1, 2, 3$), *where* $P_{l,w,j}, Q_{l,w,j} \subset \mathbb{N}$, $m_l \in \mathbb{N}\setminus\{0\}$ *and* $\widetilde{\alpha}_p^{l,w,j}, \widetilde{\beta}_q^{l,w,j}, \widetilde{\psi}_p^{l,w,j}, \widetilde{\varphi}_q^{l,w,j} \in \mathbb{R}$.

Let

$$n_1 \geq n_{\min}^1 = \max\left\{e_1 \mid e_1 \in \bigcup_{l=1}^{3}\bigcup_{w=1}^{m_l}\left(P_{l,w,1} \cup Q_{l,w,1}\right)\right\},$$

$$n_2 \geq n_{\min}^2 = \max\left\{e_2 \mid e_2 \in \bigcup_{l=1}^{3}\bigcup_{w=1}^{m_l}\left(P_{l,w,2} \cup Q_{l,w,2}\right)\right\}$$

and $r_1, r_2 \in \mathbb{N}$. If we set constants that are in the representation of surface

$$\mathbf{y}(u_1,u_2) = \begin{bmatrix} y_1(u_1,u_2) & y_2(u_1,u_2) & y_3(u_1,u_2) \end{bmatrix}^T,$$
$$y_l(u_1,u_2) \doteq y_l\left((u_1,u_2), \left\{ \left\{ \left(\alpha_p^{l,w,j}(n_j), \psi_p^{l,w,j}(n_j)\right)\right\}_{p\in P_{l,w,j}}, \left\{ \left(\beta_q^{l,w,j}(n_j), \varphi_q^{l,w,j}(n_j)\right)\right\}_{q\in Q_{l,w,j}} \right\}_{w=1,j=1}^{m_l,2}\right),$$

($l = 1, 2, 3$) in Lemma 9 as

$$\alpha_p^{l,w,1}(n_1) = \frac{\binom{2n_1}{n_1}}{\binom{2n_1}{n_1-p}} p^{r_1}\widetilde{\alpha}_p^{l,w,1}, \qquad \psi_p^{l,w,1}(n_1) \equiv \widetilde{\psi}_p^{l,w,1} + \frac{r_1\pi}{2}, \quad p \in P_{l,w,1}, \tag{19}$$

$$\beta_q^{l,w,1}(n_1) = \frac{\binom{2n_1}{n_1}}{\binom{2n_1}{n_1-q}} q^{r_1}\widetilde{\beta}_q^{l,w,1}, \qquad \varphi_q^{l,w,1}(n_1) \equiv \widetilde{\varphi}_q^{l,w,1} + \frac{r_1\pi}{2}, \quad q \in Q_{l,w,1}, \tag{20}$$

$$\alpha_p^{l,w,2}(n_2) = \frac{\binom{2n_2}{n_2}}{\binom{2n_2}{n_2-p}} p^{r_2}\widetilde{\alpha}_p^{l,w,2}, \qquad \psi_p^{l,w,2}(n_2) \equiv \widetilde{\psi}_p^{l,w,2} + \frac{r_2\pi}{2}, \quad p \in P_{l,w,2}, \tag{21}$$

$$\beta_q^{l,w,2}(n_2) = \frac{\binom{2n_2}{n_2}}{\binom{2n_2}{n_2-q}} q^{r_2}\widetilde{\beta}_q^{l,w,2}, \qquad \varphi_q^{l,w,2}(n_2) \equiv \widetilde{\varphi}_q^{l,w,2} + \frac{r_2\pi}{2}, \quad q \in Q_{l,w,2}, \tag{22}$$

($w = 1, 2, \ldots, m_l$), then for all $n_1 \geq n_{\min}^1$ and $n_2 \geq n_{\min}^2$ the cyclic surface generated by control points

$$\mathbf{d}_{i_1 i_2} = \begin{bmatrix} y_1(-i_1\lambda_{n_1}, -i_2\lambda_{n_2}) & y_2(-i_1\lambda_{n_1}, -i_2\lambda_{n_2}) & y_3(-i_1\lambda_{n_1}, -i_2\lambda_{n_2}) \end{bmatrix}^T, \tag{23}$$

($i_1 = 0, 1, \ldots, 2n_1$, $i_2 = 0, 1, \ldots, 2n_2$) describe exactly the $(r_1 + r_2)$th order mixed partial derivatives of surface $\widetilde{\mathbf{y}}$, i.e. the lth coordinate function of the cyclic surface $\mathbf{s}_{n_1,n_2}$ fulfills the equality

$$\begin{aligned} s_l(u_1,u_2) &= \frac{\partial^{r_1+r_2}}{\partial u_1^{r_1}\partial u_2^{r_2}}\widetilde{y}_l(u_1,u_2) \\ &= \sum_{w=1}^{m_l}\prod_{j=1}^{2}\left(\sum_{p\in P_{l,w,j}} \widetilde{\alpha}_p^{l,w,j} p^{r_j}\cos\left(pu_j + \widetilde{\psi}_p^{l,w,j} + \frac{r_j\pi}{2}\right) + \sum_{q\in Q_{l,w,j}} \widetilde{\beta}_q^{l,w,j} q^{r_j}\sin\left(qu_j + \widetilde{\varphi}_q^{l,w,j} + \frac{r_j\pi}{2}\right)\right), \\ &\quad \forall (u_1,u_2) \in [-2\pi, 0]\times[-2\pi,0], \ (l = 1,2,3). \end{aligned}$$

4. Closed rational trigonometric curves and surfaces

By means of cyclic basis functions (5) one can generate basis functions of type (2) as follows.

Definition 11 (*Rational Trigonometric Basis*)**.** The non-negative weight vector $\mathbf{w} = \{w_i\}_{i=0}^{2n}$ and the cyclic basis functions (5) determine the rational trigonometric basis

$$R = \left\{R_{i,n} : [-2\pi, 0] \to \mathbb{R}\right\}_{i=0}^{2n},$$

where

$$R_{i,n}(u) \doteq R_{i,n}(u; \mathbf{w}) = \frac{w_i C_{i,n}(u)}{\sum_{j=0}^{2n} w_j C_{j,n}(u)}. \tag{24}$$

Using these rational trigonometric basis functions, closed curves and surfaces can be represented as a combination of control points as formulated in the next simple definitions.

Definition 12 (*Closed Rational Trigonometric Curves*)**.** Consider the control polygon

$$\mathbf{D}_n = [\mathbf{d}_0, \mathbf{d}_1, \ldots, \mathbf{d}_{2n}] = [\mathbf{d}_i]_{i=0}^{2n} \in \mathcal{M}_{1,2n+1}\left(\mathbb{R}^\delta\right), \quad \delta \geq 2$$

and the non-negative weight vector $\mathbf{w} = \{w_i\}_{i=0}^{2n}$. The curve

$$\begin{cases} \mathbf{g}_n : [-2\pi, 0] \to \mathbb{R}^\delta \\ \mathbf{g}_n(u) \doteq \mathbf{g}_n(u; \mathbf{w}) = \sum_{i=0}^{2n} \mathbf{d}_i R_{i,n}(u; \mathbf{w}) \end{cases} \tag{25}$$

is called closed rational trigonometric curve of degree $n \geq 1$.

Definition 13 (*Closed Rational Trigonometric Surfaces*). Consider the control net

$$\mathbf{D}_{n,m} = [\mathbf{d}_{kl}]_{k=0,l=0}^{2n,2m} \in \mathcal{M}_{2n+1,2m+1}\left(\mathbb{R}^3\right)$$

and the non-negative weight vectors $\mathbf{w}^u = \{w_k^u\}_{k=0}^{2n}$ and $\mathbf{w}^v = \{w_l^v\}_{l=0}^{2m}$ in u and v directions. The surface

$$\begin{cases} \mathbf{s}_{n,m} : [-2\pi, 0] \times [-2\pi, 0] \to \mathbb{R}^3 \\ \mathbf{s}_{n,m}(u, v) \doteq \mathbf{s}_{n,m}\left(u, v; \mathbf{w}^u, \mathbf{w}^v\right) = \sum_{k=0}^{2n} \sum_{l=0}^{2m} \mathbf{d}_{kl} R_{k,n}\left(u; \mathbf{w}^u\right) R_{l,m}(v; \mathbf{w}^v) \end{cases} \tag{26}$$

is called closed rational trigonometric surface of degree (n, m).

Curves (25) inherit the properties of cyclic curves, except the cyclic symmetry, since different weights that are associated with individual control points ruin this property.

In case of rational Bézier curves – when $F_i(u)$ is the ith Bernstein polynomial in formula (2) – weights w_i and $c^i w_i$, $(0 < c \in \mathbb{R})$ determine the same curve (cf. [2]). In the forthcoming theorem we show that similar weight transformations are impossible in general for rational trigonometric curves determined by functions (24).

Theorem 14. *Weights of rational trigonometric curves of degree $n = 1$ are uniquely determined up to a positive scaling factor.*

Proof. Let us consider the closed trigonometric curve (the ellipse) determined by control points $\mathbf{d}_0, \mathbf{d}_1, \mathbf{d}_2$ and weights w_0, w_1, w_2. The evaluation of basis functions $R_{i,1}(u)$ at $\pi, \pi/3$ and $-\pi/3$ is as follows

	$R_{0,1}(u)$	$R_{1,1}(u)$	$R_{2,1}(u)$
π	0	$\frac{w_1}{w_1+w_2}$	$\frac{w_2}{w_1+w_2}$
$\frac{\pi}{3}$	$\frac{w_0}{w_0+w_2}$	0	$\frac{w_2}{w_0+w_2}$
$-\frac{\pi}{3}$	$\frac{w_0}{w_0+w_1}$	$\frac{w_1}{w_0+w_1}$	0

This means that the ellipse touches the sides $\mathbf{d}_0, \mathbf{d}_1$; $\mathbf{d}_1, \mathbf{d}_2$ and $\mathbf{d}_0, \mathbf{d}_2$ of the control triangle at the points

$$\begin{aligned} \mathbf{m}_0 &= \frac{w_0}{w_0 + w_1}\mathbf{d}_0 + \left(1 - \frac{w_0}{w_0 + w_1}\right)\mathbf{d}_1, \\ \mathbf{m}_1 &= \frac{w_1}{w_1 + w_2}\mathbf{d}_1 + \left(1 - \frac{w_1}{w_1 + w_2}\right)\mathbf{d}_2, \\ \mathbf{m}_2 &= \frac{w_0}{w_0 + w_2}\mathbf{d}_0 + \left(1 - \frac{w_0}{w_0 + w_2}\right)\mathbf{d}_2, \end{aligned} \tag{27}$$

respectively.

The pre-image of this ellipse is a cyclic curve (also an ellipse) that touches the sides of its control polygon at their midpoints. Central projection preserves incidence, therefore points of contact are mapped into points of contact. The pre-image of the segment $\mathbf{d}_0, \mathbf{d}_1$ is $\mathbf{d}_0^w, \mathbf{d}_1^w$ and the central projection of the midpoint $\mathbf{m}_0^w$ of $\mathbf{d}_0^w, \mathbf{d}_1^w$ must be $\mathbf{m}_0$. Therefore, the midpoint $\mathbf{m}_0^w$ of $\mathbf{d}_0^w, \mathbf{d}_1^w$ is uniquely determined up to similarity (cf. Fig. 1). □

5. Applications

In this section we show how rational trigonometric basis can be used for the exact description of some famous closed curves and surfaces. The process comprises the following steps:

- consider the traditional parametric form of the rational trigonometric curve/surface;
- reformulate the equations in order to apply Theorem 8/10 in the pre-image space (if this reformulation cannot be performed, then the cyclic representation does not exist in the pre-image space);
- compute control points of the cyclic representation in the pre-image space;

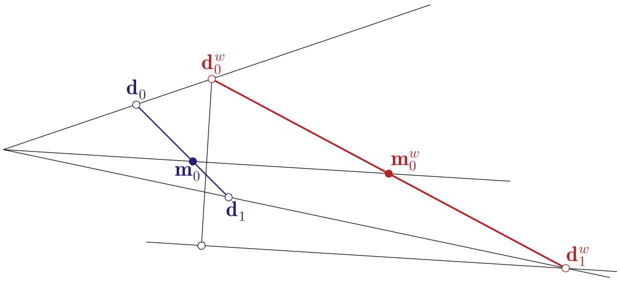

Fig. 1. Construction of the pre-image of the midpoint.

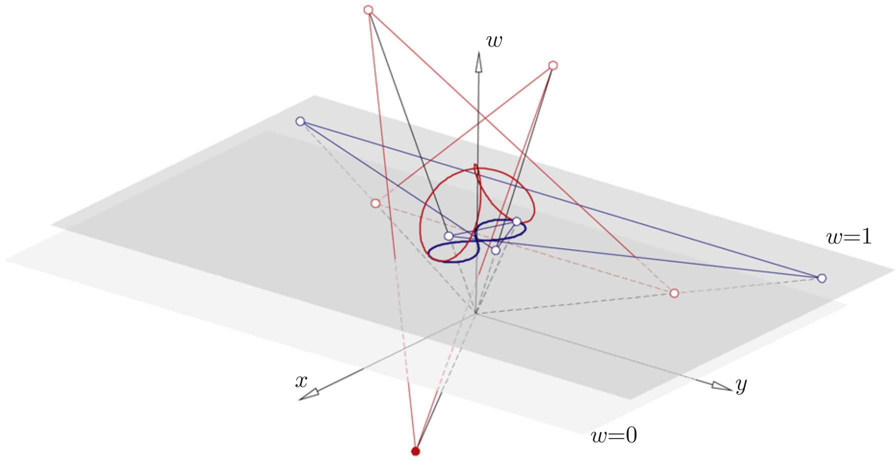

Fig. 2. The exact description of Bernoulli's lemniscate by rational trigonometric curve of degree 2 (blue) and its cyclic pre-image (red). The w coordinate of the filled red control point is negative. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

- project control points onto the hyperplane $w = 1$ that results the control points and weights for the rational representation.

This generation process, however does not guarantee the positivity of weights, since the last coordinate of some control points in the pre-image space can be negative. The object we want to describe is closed consisting of one branch, thus its pre-image must not intersect the vanishing plane ($w = 0$), i.e. the w coordinate of all of its points are of the same sign. The increase of the degree of cyclic curves/surfaces results a sequence of control polygons/nets that converges to the curve/surface (cf. Proposition 7 and Subsection 4.1 of [12]). Therefore, it is guaranteed that exists such a finite and minimal degree for which all weights are positive. This is illustrated in Figs. 2 and 3.

5.1. Lemniscate of Bernoulli

The parametric form of Bernoulli's lemniscate is

$$x(u) = \frac{\cos u}{1+\sin^2 u},$$
$$y(u) = \frac{\sin u \cos u}{1+\sin^2 u},$$
$$u \in [0, 2\pi].$$

The corresponding curve in the pre-image space is

$$\widetilde{x}_1(u) = \cos u,$$
$$\widetilde{x}_2(u) = \frac{\sin 2u}{2},$$
$$\widetilde{x}_3(u) = \frac{3-\cos 2u}{2}.$$

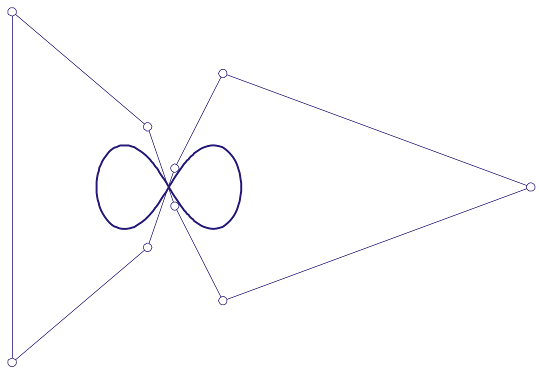

Fig. 3. Exact description of Bernoulli's lemniscate with a rational trigonometric curve of degree 4; all weights are positive.

If $n \geq 2$ control points of the cyclic curve, that exactly describes the above curve in the pre-image space, are on the curve

$$x_1(u) = \frac{n+1}{n}\cos u,$$
$$x_2(u) = \frac{n^2+3n+2}{2n(n-1)}\sin 2u,$$
$$x_3(u) = \frac{3}{2} - \frac{n^2+3n+2}{2n(n-1)}\cos 2u,$$

at parameter values $-i\lambda_n$ $(i = 0, 1, \ldots, 2n)$, i.e.

$$\left[\mathbf{d}_i^w\right]_{i=0}^{2n} = \left[\left[x_1(-i\lambda_n) \quad x_2(-i\lambda_n) \quad x_3(-i\lambda_n)\right]^T\right]_{i=0}^{2n}.$$

Projecting these control points from the origin onto the hyperplane $w = 1$, we obtain the control points $\mathbf{d}_i$ and weights w_i of the rational representation in the form

$$[\mathbf{d}_i]_{i=0}^{2n} = \left[\left[\frac{x_1(-i\lambda_n)}{x_3(-i\lambda_n)} \quad \frac{x_2(-i\lambda_n)}{x_3(-i\lambda_n)}\right]^T\right]_{i=0}^{2n}, \qquad [w_i]_{i=0}^{2n} = [x_3(-i\lambda_n)]_{i=0}^{2n}. \tag{28}$$

In this way we can describe the whole curve by means of a single rational trigonometric curve of degree two. Using complex rational Bézier curves, proposed in [23], just an arc of the Bernoulli's lemniscate can exactly be described. Fig. 2 illustrates the projection concept: the blue curve and control polygon is the rational cyclic representation of the lemniscate and the red ones are their counterpart in the pre-image space. One can also observe that the w coordinate of the filled red control point is negative, consequently the weight of the corresponding blue control point is negative, therefore the blue curve is outside the convex hull of its control polygon. However, if the degree is ≥ 4, all weights become positive as it is shown in Fig. 3.

5.2. Zhukovsky profile

The conformal mapping

$$z + \frac{1}{z}$$

in the complex plane is called Zhukovsky transform (also written Joukowski or Joukowsky) and is used in aerodynamics. The basic form of the airfoil profile is the map of a circle

$$(x - c_x)^2 + \left(y - c_y\right)^2 = r^2$$

that passes through the point $(1, 0)$ and contains the point $(-1, 0)$. The resulted parametric form of its Zhukovsky transform is

$$r\cos u + c_x + \frac{r\cos u + c_x}{r^2 + 2r\left(c_x\cos u + c_y\sin u\right) + c_x^2 + c_y^2} + \mathrm{i}\left(r\sin u + c_y - \frac{r\sin u + c_y}{r^2 + 2r\left(c_x\cos u + c_y\sin u\right) + c_x^2 + c_y^2}\right)$$

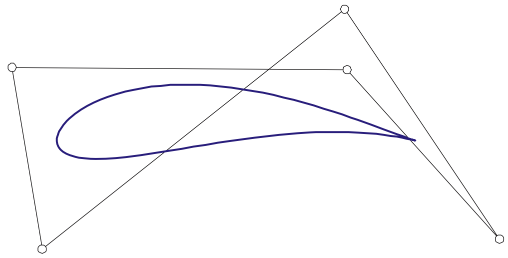

Fig. 4. Zhukovsky airfoil profile with settings $n = 2$, $r = 1.1$ and $v = 0.1$.

from which we obtain the coordinate functions

$$\begin{cases} x(u) = (r\cos u + c_x)\left(1 + \dfrac{1}{r^2 + 2r\left(c_x\cos u + c_y\sin u\right) + c_x^2 + c_y^2}\right) \\ y(u) = \left(r\sin u + c_y\right)\left(1 - \dfrac{1}{r^2 + 2r\left(c_x\cos u + c_y\sin u\right) + c_x^2 + c_y^2}\right), \end{cases}$$
$$1 < \sqrt{r^2 - c_y^2},\, c_x = 1 \mp \sqrt{r^2 - c_y^2}, \quad u \in [0, 2\pi].$$

In the pre-image space control points have to be located on the curve

$$\begin{aligned} x_1(u) &= c_x\left(2r^2 + c_x^2 + c_y^2 + 1\right) + r\left(r^2 + 3c_x^2 + c_y^2 + 1\right)\frac{n+1}{n}\cos u + r^2 c_x \frac{n^2 + 3n + 2}{n(n-1)}\cos 2u \\ &\quad + 2rc_x c_y \frac{n+1}{n}\sin u + r^2 c_y \frac{n^2+3n+2}{n(n-1)}\sin 2u, \\ x_2(u) &= c_y\left(2r^2 + c_x^2 + c_y^2 - 1\right) + 2rc_x c_y\frac{n+1}{n}\cos u - r^2 c_y\frac{n^2+3n+2}{n(n-1)}\cos 2u \\ &\quad + r\left(r^2 + c_x^2 + 3c_y^2 - 1\right)\frac{n+1}{n}\sin u + r^2 c_x\frac{n^2+3n+2}{n(n-1)}\sin 2u, \\ x_3(u) &= \left(r^2 + c_x^2 + c_y^2\right) + 2rc_x\frac{n+1}{n}\cos u + 2rc_y\frac{n+1}{n}\sin u \end{aligned}$$

using formula (28). The control point based representation of the Zhukovsky profile is illustrated in Fig. 4.

5.3. Boy surface

The surface

$$\begin{bmatrix} \dfrac{\widetilde{y}_1(u,v)}{\widetilde{y}_4(u,v)} & \dfrac{\widetilde{y}_2(u,v)}{\widetilde{y}_4(u,v)} & \dfrac{\widetilde{y}_3(u,v)}{\widetilde{y}_4(u,v)} \end{bmatrix}^T, \quad (u,v) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times [0, \pi] \tag{29}$$

is a homotopy (smooth transition) between the Roman and Boy surfaces, where

$$\begin{aligned} \widetilde{y}_1(u,v) &= \sqrt{2}\cos 2u\cos^2 v + \cos u\sin 2v \\ &= \frac{\sqrt{2}}{2}\cos 2u + \frac{\sqrt{2}}{2}\cos 2u\cos 2v + \cos u\sin 2v, \\ \widetilde{y}_2(u,v) &= \sqrt{2}\sin 2u\cos^2 v - \sin u\sin 2v \\ &= \frac{\sqrt{2}}{2}\sin 2u + \frac{\sqrt{2}}{2}\sin 2u\cos 2v - \sin u\sin 2v, \\ \widetilde{y}_3(u,v) &= 3\cos^2 v \\ &= \frac{3}{2}(1 + \cos 2v), \\ \widetilde{y}_4(u,v) &= 2 - \alpha\sqrt{2}\sin 3u\sin 2v. \end{aligned}$$

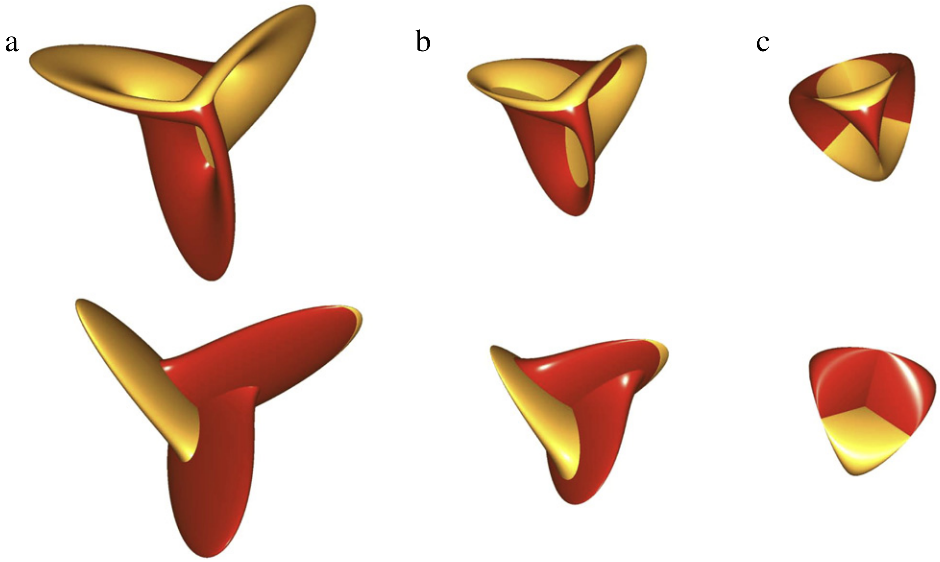

Fig. 5. Different special cases of the surface (29): (a) the Boy surface ($\alpha = 1$); (b) a smooth transition between the Boy and the Roman surface of Steiner ($\alpha = 0.7$); (c) the Roman surface of Steiner ($\alpha = 0$).

This surface in the pre-image space can exactly be described by means of a cyclic surface of degree ($n \geq 3, m \geq 2$) the control points of which are located on the surface

$$\begin{aligned}
y_1(u,v) &= \frac{\sqrt{2}}{2}\frac{n^2+3n+2}{n(n-1)}\cos 2u + \frac{\sqrt{2}}{2}\frac{n^2+3n+2}{n(n-1)}\frac{m^2+3m+2}{m(m-1)}\cos 2u\cos 2v \\
&\quad + \frac{n+1}{n}\frac{m^2+3m+2}{m(m-1)}\cos u\sin 2v, \\
y_2(u,v) &= \frac{\sqrt{2}}{2}\frac{n^2+3n+2}{n(n-1)}\sin 2u + \frac{\sqrt{2}}{2}\frac{n^2+3n+2}{n(n-1)}\frac{m^2+3m+2}{m(m-1)}\sin 2u\cos 2v, \\
&\quad - \frac{n+1}{n}\frac{m^2+3m+2}{m(m-1)}\sin u\sin 2v \\
y_3(u,v) &= \frac{3}{2}\left(1+\frac{m^2+3m+2}{m(m-1)}\cos 2v\right), \\
y_4(u,v) &= 2-\alpha\sqrt{2}\frac{n^3+6n^2+11n+6}{n\left(n^2-3n+2\right)}\frac{m^2+3m+2}{m(m-1)}\sin 3u\sin 2v
\end{aligned}$$

at equally spaced parameter values, i.e.,

$$\left[\mathbf{d}_{ij}^{w}\right]_{i=0,j=0}^{2n,2m} = \left[\left[y_1(-i\lambda_n,-j\lambda_m) \quad y_2(-i\lambda_n,-j\lambda_m) \quad y_3(-i\lambda_n,-j\lambda_m) \quad y_4(-i\lambda_n,-j\lambda_m)\right]^T\right]_{i=0,j=0}^{2n,2m}.$$

Therefore, the control points of the rational representation are

$$\begin{aligned}
\left[\mathbf{d}_{ij}\right]_{i=0,j=0}^{2n,2m} &= \left[\left[\frac{y_1(-i\lambda_n,-j\lambda_m)}{y_4(-i\lambda_n,-j\lambda_m)} \quad \frac{y_2(-i\lambda_n,-j\lambda_m)}{y_4(-i\lambda_n,-j\lambda_m)} \quad \frac{y_3(-i\lambda_n,-j\lambda_m)}{y_4(-i\lambda_n,-j\lambda_m)}\right]^T\right]_{i=0,j=0}^{2n,2m}, \\
\left[w_{ij}\right]_{i=0,j=0}^{2n,2m} &= \left[y_4(-i\lambda_n,-j\lambda_m)\right]_{i=0,j=0}^{2n,2m}.
\end{aligned} \tag{30}$$

Based on the proposed method, Fig. 5 shows some special cases of the described homotopy between the Boy and Roman surfaces.

5.4. Dupin cyclide

The parametric form of Dupin cyclide is

$$x(u,v) = \frac{\mu(c-a\cos u\cos v)+b^2\cos u}{a-c\cos u\cos v},$$

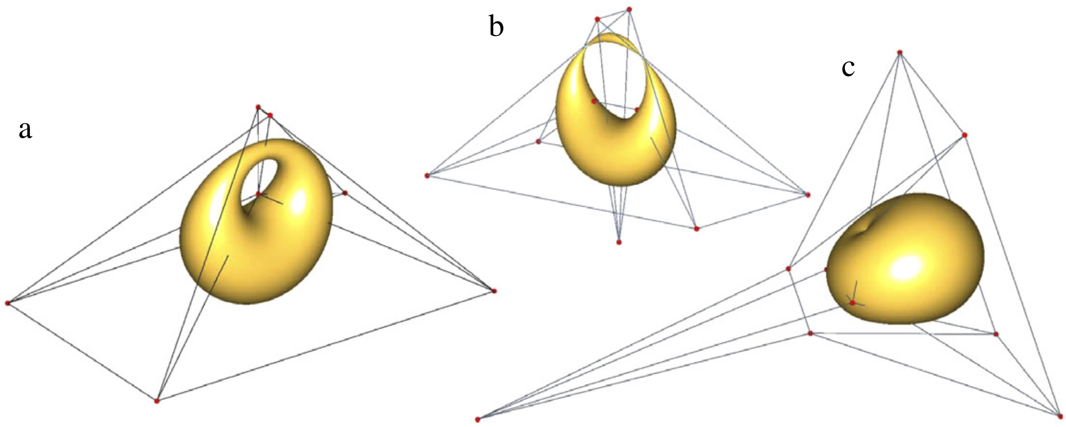

Fig. 6. The three different basic cases of Dupin's cyclide ($a = 6$, $b = 4\sqrt{2}$, $c = 2$): (a) ring ($\mu = 3$); (b) horn ($\mu = 1.5$); and (c) spindle ($\mu = 12$).

$$
\begin{aligned}
y(u, v) &= \frac{b \sin u \,(a - \mu \cos v)}{a - c \cos u \cos v},\\
z(u, v) &= \frac{b \sin v \,(c \cos u - \mu)}{a - c \cos u \cos v},\\
(u, v) &\in [0, 2\pi] \times [0, 2\pi],
\end{aligned}
$$

where parameters a, b and c have to fulfill the conditions $c^2 = a^2 - b^2$ and $a \geq c$. Values of a, c and μ determine the type of the cyclide:

- when $c < \mu \leq a$ it is a ring cyclide;
- when $0 < \mu \leq c$ it is a horned cyclide;
- when $\mu > a$ it is a spindle cyclide.

In order to represent this surface in rational trigonometric bases (24), control points in the pre-image space have to be located on the surface

$$
\begin{aligned}
y_1(u, v) &= \mu \left(c - a \frac{n+1}{n} \frac{m+1}{m} \cos u \cos v \right) + b^2 \frac{n+1}{n} \cos u,\\
y_2(u, v) &= b \frac{n+1}{n} \sin u \left(a - \mu \frac{m+1}{m} \cos v \right),\\
y_3(u, v) &= b \frac{m+1}{m} \sin v \left(c \frac{n+1}{n} \cos u - \mu \right),\\
y_4(u, v) &= a - c \frac{n+1}{n} \frac{m+1}{m} \cos u \cos v,
\end{aligned}
$$

where $n \geq 1$ and $m \geq 1$ are natural numbers. Control points of the rational representation can be computed by formula (30). Fig. 6 presents three different Dupin cyclides along with their control net.

6. Conclusions

Nowadays, in CAGD curves and surfaces are described by the combination of control points and basis functions. If we associate weights with basis functions (or control points) and normalize them by their weighted sum, we obtain another system of basis functions that we call quotient basis. Based on the fact that curves/surfaces defined by such quotient basis can also be obtained by the central projection of a corresponding object defined by the original basis functions, we show some common characteristics of them. Using this concept, we specify the rational counterpart of the recently introduced cyclic basis. This extension provides a powerful tool for the exact control point based representation of a class of closed rational trigonometric curves and surfaces. The applicability of the proposed tool is illustrated via several examples, such as lemniscate of Bernoulli, Zhukovsky airfoil profile, Dupin cyclide and Boy surface. We wish to emphasize, that using other rational curve and surface description methods, such as the rational complex Bézier curves/surfaces, only arcs/patches of a rational trigonometric object can be modeled exactly and these arcs/patches have to fulfill heavy geometric constraints along the joints, in order to model the whole object.

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